Radiation Pressure Cooling and Quantum Non Demolition

G. Cella

Prepared for the 4th VIRGO-EGO-SIGRAV SCHOOL ON GRAVITATIONAL WAVES
Cascina May 23rd - 27th, 2005

version of May 23, 2006
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Introduction

Gravitational waves interact with matter very weakly. For this reason the response of the detector to them can be comparable with its quantum uncertainties. It is important, in the perspective of future improvements of gravitational wave detectors, to understand the issue of quantum noise and of its reduction.

These are preliminary notes which can be used as a background for the lessons of the school. They are not yet complete, and I hope that they will evolve toward a more consistent and complete introduction to the subject.

The more up to date version can be found on [http://www.df.unipi.it/cella/QND/](http://www.df.unipi.it/cella/QND/). Corrections and suggestions are welcome.

Solutions of the exercises can be found at the end of the document.

1.1 Quantum measurements

As a starting point we remind the von Neumann’s postulate of reduction, which gives us informations about a so called ideal measurement. Suppose that we have a system described by a given density operator $\hat{\rho}$.

1. If we measure the observable $\hat{q}$ we will obtain one of its eigenvalues $q_n$, with a probability given by

$$w_n = \text{Tr} (|q_n><q_n|\hat{\rho}) .$$

2. After the measurement the system will be in the state

$$\hat{\rho}' = \frac{1}{w_n}|q_n><q_n|\hat{\rho}|q_n><q_n| .$$

We are interested in describing a less idealized measurement. For example, in a real detector there is always a finite resolution. Suppose that the measurement apparatus can give us informations not about the exact value of the observable $\hat{q}$, but only about the fact that this is contained inside a given range $I_j$. Taking for definiteness $I_j = [q_j, q_{j+1}]$ We can obtain a complete description of the apparatus introducing a set of projectors

$$\hat{E}_j = \int_{I_j} |q><q|dq$$

which is required to be complete

$$\sum_j \hat{E}_j = \hat{I}$$

because the sum of the probabilities for all possible results of the measurements is unity. The set is also required to be orthogonal

$$\hat{E}_j \hat{E}_k = \delta_{jk} \hat{E}_j$$
because if we prepare our system in a state with a well defined value of the observable we want to obtain a well defined result from an immediate measurement.

Now we can generalize the previous postulates:

1. The probability of finding the value of the observable inside \( I_j \) is given by

\[
    w_j = \text{Tr} \left( \hat{E}_j \hat{\rho} \right) .
\]  

(1.1)

2. After the measurement the system will be in the state

\[
    \hat{\rho}' = \frac{1}{w_j} \hat{E}_j \hat{\rho} \hat{E}_j .
\]  

(1.2)

Notice that, as we expected, this ensure complete repeatability. If we make a second measure immediately after the first we find the system in the same interval as before with probability one.

The described generalization (the orthogonal measurements case) does not take into account the fact that the measurement apparatus can respond in a not completely deterministic way. In order to deal with this we introduce a set of conditional probabilities \( w(q' \mid q) \) which give us the probability that the apparatus report the value \( q' \) when the system is in the status \( | q > \). We want to obtain a further generalization of our two postulates.

For the first point, we need to evaluate the probability of finding the value \( q' \) after a measurement. This can be written of course

\[
    w(q') = \int w(q' \mid q) \text{Tr}(|q> <q| \hat{\rho}) \, dq
\]

and if we introduce the operators

\[
    \hat{W}(q') = \int |q > w(q' \mid q) <q| \, dq
\]

which are a complete set

\[
    \int \hat{W}(q') dq' = \hat{I}
\]

we obtain

\[
    w(q') = \text{Tr} \left( \hat{W}(q') \hat{\rho} \right) .
\]  

(1.3)

This looks like Eq. (1.1). The operators \( \hat{W} \) are not projectors in the general case

\[
    \hat{W}(q_1) \hat{W}(q_2) = \int |q > w(q_1 \mid q) w(q_2 \mid q) <q| \, dq
\]

however they commute with each other

\[
    [\hat{W}(q), \hat{W}(q')] = 0 .
\]

Any self commuting decomposition of the unity describes a measurement of a single observable, and if we take

\[
    w(q', q) = \sum_j \delta(q' - q_j) I_j(q)
\]

where

\[
    I_j(q) = \begin{cases} 
        1 & \text{if } q_j \leq q < q_{j+1} \\
        0 & \text{otherwise}
    \end{cases}
\]
we recover the orthogonal measurements case.

The second point concerns the description of the system after a measurement. We suppose that the new density matrix could be written as

\[
\hat{\rho}(q') = \frac{1}{w(q')} \hat{\Omega}(q') \hat{\Omega}^+(q')
\]

(1.4)

where \(\hat{\Omega}\) is an operator which must be determined. Taking the trace we get

\[
1 = \text{Tr}(\hat{\rho}(q')) = \frac{1}{w(q')} \text{Tr}\left( \hat{\Omega}^+(q') \hat{\Omega}(q') \hat{\rho} \right)
\]

and comparing with Eq. (1.3) we find the result

\[
\hat{W}(q') = \hat{\Omega}^+(q') \hat{\Omega}(q')
\]

(1.5)

which tell us that \(\hat{\Omega}\) can be represented as the product

\[
\hat{\Omega}(q') = \hat{U}(q') \hat{W}^{1/2}(q')
\]

(1.6)

where \(\hat{U}\) is unitary and

\[
\hat{W}^{1/2}(q') = \int |q > \sqrt{w(q'|q)} < q| dq.
\]

(1.7)

Note that this operator depends only by the conditional probabilities which describe our detector. The unitary operator \(\hat{U}\) on the contrary depends by the details of the measurement procedure.

### 1.1.1 Heisenberg uncertainty relations

A fundamental consequence of the non commutative character of observables in quantum mechanics are the well known Heisenberg uncertainty relations. Given two Hermitian operators \(\hat{A}, \hat{B}\) we define

\[
\delta\hat{A} = \hat{A} - \langle \hat{A} \rangle
\]

\[
\delta\hat{B} = \hat{B} - \langle \hat{B} \rangle
\]

and we put

\[
\hat{W} = \delta\hat{A} + \lambda e^{i\theta} \delta\hat{B}.
\]

From the positivity condition for

\[
\left\langle \hat{W} \hat{W}^+ \right\rangle = \left\langle \delta\hat{A}^2 \right\rangle + \lambda \left( \cos \theta \left\langle [\delta\hat{A}, \delta\hat{B}]_+ \right\rangle - i \sin \theta \left\langle [\delta\hat{A}, \delta\hat{B}] \right\rangle \right) + \lambda^2 \left\langle \delta\hat{B}^2 \right\rangle \geq 0
\]

we obtain

\[
\left( \cos \theta \left\langle [\delta\hat{A}, \delta\hat{B}]_+ \right\rangle - i \sin \theta \left\langle [\delta\hat{A}, \delta\hat{B}] \right\rangle \right)^2 \leq 4 \left\langle \delta\hat{A}^2 \right\rangle \left\langle \delta\hat{B}^2 \right\rangle
\]

and maximizing over \(\theta\)

\[
\left\langle \delta\hat{A}^2 \right\rangle \left\langle \delta\hat{B}^2 \right\rangle \geq \frac{1}{4} \left\langle [\delta\hat{A}, \delta\hat{B}] \right\rangle^2 + \frac{1}{4} \left\langle [\delta\hat{A}, \delta\hat{B}]_+ \right\rangle^2.
\]

The second piece on the left is the square of the expectation value of the anti commutator of the two observables. This can be different from zero also in the classical limit, when it reduces to the statistical covariance of the two quantities. It express the fact that the product of the statistical uncertainties of the two observables cannot be zero if these are correlated.
The first piece is entirely non classical, because it vanishes when the two operators commute. Even when two non commuting observables are uncorrelated the product of their variances cannot be zero, which express the well known fact that it is impossible to measure both of them with arbitrary precision.

For two canonically conjugate quantities, as the position and the momentum, we obtain
\[ \Delta x \Delta p \geq \sqrt{\frac{1}{4} \hbar^2 + \sigma_{xp}^2} \]

where
\[ \sigma_{xp} = \frac{1}{2} \langle (\hat{x} - \langle \hat{x} \rangle)(\hat{p} - \langle \hat{p} \rangle) + (\hat{p} - \langle \hat{p} \rangle)(\hat{x} - \langle \hat{x} \rangle) \rangle \]
is the (quantum) covariance and
\[ \Delta O = \sqrt{\left\langle (\hat{O} - \langle \hat{O} \rangle)^2 \right\rangle} \]

Note that this results is true both for a pure state and for a system described by a density operator. In fact we never specify the details of how the expectation value is obtained, using only its linearity. For a system described by a density operator we have of course
\[ \langle \hat{O} \rangle = \text{Tr} \left( \hat{O} \hat{\rho} \right) . \]

1.1.2 Repeated measurements

We give now a very simple analysis of a repeated measurement of an observable. We will avoid to specify details about the measurement process, by using only the Heisenberg uncertainty relation. Our target is to monitor a free particle. This system is described by the Hamiltonian
\[ \hat{H} = \frac{1}{2m} \hat{\sigma}^2 \]

and the Heisenberg equations of motion can be written as
\[
\begin{align*}
\dot{\hat{p}} &= \frac{1}{i\hbar} [\hat{p}, \hat{H}] = 0 \\
\dot{\hat{x}} &= \frac{1}{i\hbar} [\hat{x}, \hat{H}] = \frac{1}{m} \hat{p}
\end{align*}
\]

which are easily integrated between \( t_n \) and \( t_{n+1} = t_n + \tau \) to give
\[
\begin{align*}
\hat{p}(t_{n+1}) &= \hat{p}(t_n) \\
\hat{x}(t_{n+1}) &= \hat{x}(t_n) + \frac{\tau}{2m} \hat{p}(t_n) .
\end{align*}
\]

Suppose now that for each \( n \) we make a measurement of the position \( \hat{x} \). The value of \( \hat{x} \) at \( t = t_n \) is not completely determined, and this indetermination \( \Delta x_n \) depends on the past evolution of the system. It is in principle now possible to measure the position with an arbitrary precision \( \Delta x_n \). However immediately after the measurement there will be a lower bounds over the indetermination \( \Delta p_n \) of our system given by
\[ \Delta p_n^2 \geq \frac{\hbar^2 + \langle (\delta \hat{x} \delta \hat{p} + \delta \hat{p} \delta \hat{x})^2 \rangle}{4 \Delta x_n^2} . \]

In the best case we suppose that it is possible to make a measurement without introducing correlations between the position and the momenta. If we use the previous
Quantum measurements

equation of motion we can write the evolution of the indetermination as

\[ \Delta p_{n+1}^2 = \Delta p_n^2 \]

\[ \Delta x_{n+1}^2 = \Delta x_n^2 + \frac{\tau^2}{4m^2} \Delta p_n^2 \]

and using the Heisenberg bound we get

\[ \Delta x_{n+1}^2 = \Delta x_n^2 + \frac{\tau^2 \hbar^2}{16m^2 \Delta p_n^2} \]

which is minimized by choosing

\[ \Delta x_n^2 = \frac{\tau \hbar}{4m}, \quad \Delta p_n^2 = \frac{\hbar m}{\tau}. \]

In this case

\[ \Delta x_n = \sqrt{\frac{\tau \hbar}{2m}}, \quad \Delta p_n = \sqrt{\frac{\hbar m}{\tau}} \quad (1.10) \]

where the product \( \Delta x_n \Delta p_n \) is a factor \( \sqrt{2} \) larger that the minimum value admitted by the Heisenberg principle.

The final message is that in order to minimize the errors in a series of repeated measurements it is better to avoid to measure the observable with an arbitrary precision. This imply that there is an optimal error on the observable, which is given by Eq. (1.10), and it is generally called Standard Quantum Limit.

1.1.3 Monitoring a classical force

A more interesting example is the use of a quantum harmonic oscillator coupled to a classical unknown force \( F(t) \). This time the target is to obtain informations about \( F(t) \) using the results of repeated measurement done on the quantum system. The Hamiltonian is in this case

\[ \hat{H} = \frac{1}{2m} \hat{p}^2 + \frac{m\omega^2}{2} \hat{x}^2 - F(t) \hat{x} \]

and the Heisenberg equation of motions

\[ \dot{\hat{p}} = \frac{\hbar}{i} [\hat{p}, \hat{H}] = -m\omega^2 \hat{x} + F(t) \]

\[ \dot{\hat{x}} = \frac{\hbar}{i} [\hat{x}, \hat{H}] = \frac{1}{m} \hat{p} \]

can be easily solved introducing the operator \( \hat{q} = \hat{p} + i m \omega \hat{x} \). We obtain a simple equation of motion

\[ \dot{\hat{q}} = i \omega \hat{q} + F(t) \]

which can be solved obtaining

\[ \hat{p}(t_{n+1}) = \hat{p}(t_n) \cos \omega \tau - m \omega \hat{x}(t_n) \sin \omega \tau + \int_{t_n}^{t_{n+1}} F(t') \cos \omega (t_{n+1} - t') dt' \]

\[ \hat{x}(t_{n+1}) = \frac{1}{m \omega} \hat{p}(t_n) \sin \omega \tau + \hat{x}(t_n) \cos \omega \tau + \frac{1}{m \omega} \int_{t_n}^{t_{n+1}} F(t') \sin \omega (t_{n+1} - t') dt'. \]

These equations tell us that we can obtain informations about the classical force preparing the oscillator in some given state at the time \( t = t_n \), and measuring it at \( t = t_{n+1} \).
Let us analyze now the precision we can obtain in the monitoring of the force with a coordinate measurement. The first step is to use the previous equations to evaluate the coordinate indetermination. Neglecting again the correlations between position and momenta we can write
\[
<\delta x(t_{n+1})^2> = \left(\frac{\sin \omega \tau}{m\omega}\right)^2 <\delta p(t_n)^2> + \cos^2 \omega \tau <\delta x(t_n)^2>
\]
\[= \left(\frac{\sin \omega \tau}{m\omega}\right)^2 \frac{\hbar^2}{4 <\delta x(t_n)^2>} + \cos^2 \omega \tau <\delta x(t_n)^2>.
\]
From this relation we can see that the situation is similar to the one discussed in the previous subsection. If we increase the precision of the coordinate measurement at \(t_n\) we increase the indetermination in \(\hat{p}\), so that the precision in the next measurement is reduced. There is however a particular case which is different. If we make measurements separated by intervals \(\tau = \pi/\omega\), which is half the period of the oscillator, we cancel the first term obtaining
\[
<\delta x(t_{n+1})^2> =<\delta x(t_n)^2>
\]
and at this point we can increase the precision of the coordinate measurement \(<\delta x(t_n)^2> \rightarrow 0\) without consequences. This means that we are able to perform repeated measurements of arbitrary precision. The intuitive explanation is that an harmonic oscillators return to its original position after each period (and in the opposite one after half a period) independently from the initial value of \(\hat{p}\). This kind of strategy is called stroboscopic measurement.

1.4 Quantum non demolition measurements

In the last section we learned, from a very simple example, that there is a lower bound on the uncertainty of a series of repeated measurement of the coordinate. The reason for the existence of this Standard Quantum Limit is connected to the fact that when we measure the coordinate of our system we introduce an unpredictable perturbation of its momentum. This perturbation is transferred to the coordinate by the successive temporal evolution (see Eq. (1.9)), in other words our measurement generate a “back action” on the system.

To obtain a characterization of a QND observable we come back to the description of the measurement given in Section 1.1. Formally we can describe the measure as a two step process, the first one being
\[
\hat{\rho}(q) \rightarrow \frac{1}{w(q)} W^{1/2}(q) \hat{\rho}(q) W^{1/2}(q)
\]
followed by the unitary evolution defined by the operator \(\hat{U}\) (see Eq. (1.6) and (1.4)). An important observation is that the first step cannot describe the back action of the measurement, because the value of the measured variable does not change (the operator \(\hat{W}\) commutes with the observable \(\hat{q}\)). It follows that the effect of the back action should be contained in \(\hat{U}\), and if \(\hat{U}\) commutes also with \(\hat{q}\) we will have a QND measurement.
Quantum optics

In these notes we are interested in an understanding of how the quantum fluctuations of light manifest themselves in an optical apparatus, which will be at the end an interferometric detector of gravitational waves. In the typical situation we are interested to a beam of light propagate freely between simple optical devices as mirrors, lenses and beam splitters which act on it changing its direction of propagation and/or splitting and recombining it. Considerable care must be taken in order to limit as much as possible the number of modes of the electromagnetic field which are coupled by the apparatus. In the ideal case a single, well defined mode describe the propagation of the beam along the optical axis between optical elements. More generally the apparatus can be represented by a directed graph (see for example Figure 2.8 on page 28) where an arrow correspond to free propagating electromagnetic modes and the vertices optical elements which act linearly on them.

The free propagating electromagnetic field of a beam is described in the most appropriate way by a Gaussian wave packet which moves and spread transversely at the same time, and a well developed technology exist for do that. However here we are not interested in this kind of details, and we want to concentrate on the basic effects of quantizations. For this reason we will introduce a simplified model which deals with the free propagation using a one dimensional quantum model: this is done in Section 2.1.

The modifications induced on a beam by the kind of optical devices we are interested to can be described by linear operators which act on “input modes” and transform them in combination of “output modes”. The consistency conditions these operators are subject to and the prototype of them (the beam splitter) are discussed in Section 2.2 and used to describe some simple examples, like a resonant cavity and a Michelson interferometer.

2.1 Electromagnetic fields

In this section we will introduce a simplified model for the quantized electromagnetic field. The main simplification in the model will be its one dimensional character: we will suppose that the fields depend only on a single coordinate, that we use to model the propagation of our “beam” along the optical axis. We must remember that in this way we hide under the carpet a number of details which are required for more quantitative investigations. For example, a real laser beam is limited in the transverse direction. The transverse section evolves in a non trivial way, and its correct description should be based on the theory of Gaussian beams.

However this is not an obstacle to understand the issues we are interested to, and it is better to avoid unnecessary details that could distract from the main point. So we
start rewriting the Maxwell’s equations in presence of dielectric materials as

\[ \vec{\nabla} \cdot \vec{D} = 0 \quad \rightarrow \quad \frac{\partial D_x}{\partial x} = 0 \]
\[ \vec{\nabla} \cdot \vec{B} = 0 \quad \rightarrow \quad \frac{\partial B_x}{\partial x} = 0 \]
\[ \vec{\nabla} \wedge \vec{H} = \frac{\partial \vec{D}}{\partial t} \quad \rightarrow \quad \frac{\partial D_x}{\partial t} = 0, \frac{\partial H_y}{\partial x} + \frac{\partial D_y}{\partial t} = 0, \frac{\partial H_z}{\partial x} - \frac{\partial D_z}{\partial t} = 0 \]
\[ \vec{\nabla} \wedge \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \rightarrow \quad \frac{\partial B_x}{\partial t} = 0, \frac{\partial E_z}{\partial x} - \frac{\partial B_z}{\partial t} = 0, \frac{\partial E_y}{\partial x} + \frac{\partial B_y}{\partial t} = 0 \]

where we assumed the independence of the fields from the spatial coordinates \( y \) and \( z \), \( \vec{B} = \mu \vec{H} \) and \( \vec{D} = \epsilon \vec{E} \). In the general case \( \mu \) and \( \epsilon \) can be functions of the position and also of the time. We will assume that both \( \mu \) and \( \epsilon \) can vary only in the \( x \) spatial direction, and also that the wave we are interested to propagate only in the same direction.

To avoid mathematical subtleties we limit the space to a finite volume with transverse section \( S \) and length \( L \), and we impose periodic boundary conditions. At the end we will take the limit \( L \to \infty \), while \( S \) can be interpreted as the transverse “beam” section. We see that \( D_x \) and \( B_x \) are two constant fields which are not relevant for our purposes, and that the two sets \((B_z, E_y)\) and \((B_y, -E_z)\) are uncoupled: they correspond to the two different polarization degrees of freedom.

Fixing our attention on a particular polarization, for example the \((B_z, E_y)\) set, we can derive the fields from a scalar potential \( A \) as

\[ B_z = \frac{\partial A}{\partial x} \]
\[ E_y = -\frac{\partial A}{\partial t} \]

so that one of the two equations of motion is automatically satisfied and the second one becomes

\[ \frac{\partial}{\partial x} \left( \frac{1}{\mu} \frac{\partial A}{\partial x} \right) - \frac{1}{c^2} \frac{\partial}{\partial t} \left( \frac{\partial A}{\partial t} \right) = 0 \tag{2.1} \]

which can also be obtained starting from the Lagrangian density

\[ \mathcal{L} = \frac{S \epsilon_0}{2} \left( \epsilon E_y^2 - \frac{c^2}{\mu} B_z^2 \right) \]

The Equation (2.1) looks like a wave equation, but it is a bit more complicated because we are allowing the constants which describe the linear response of the medium to the electromagnetic field to depend on the spatial coordinate \( x \) and on the time. We suppose however that at positive and negative spatial infinity \( \mu, \epsilon \to 1 \), so that in these asymptotic regions we recover the usual wave equation in the vacuum. In this way we can classify the solutions of the general equation using their asymptotic behavior, which as we know can be a plane wave

\[ A \sim e^{-i\omega t + ikx} \]

with a given wave vector \( k = \pm \omega/c \).

### 2.1.1 Quantization

Given two solutions of the Eq. 2.1 it is possible to define a scalar product

\[ (A_1, A_2) = \frac{i \epsilon_0 S}{\hbar} \int \epsilon \left( A_1^\ast \frac{\partial A_2}{\partial t} - A_2 \frac{\partial A_1^\ast}{\partial t} \right) dx \]
which is conserved by the time evolution. Note that \(< A_1, A_2 >^* = < A_2, A_1 >\) and
\(< A_1^*, A_2^* > = - < A_2, A_1 >\), so that \(< A_1, A_2 >^* = - < A_1^*, A_2^* >\).

**Exercise 1** Verify that the scalar product is in fact conserved.

We will use this scalar product (which is not positive definite) to define a norm, and the creation and destruction operators associated with a given set of positive norm modes.

Following the usual canonical quantization procedure we find the momentum conjugate to the field \(A(x)\)
\[
\Pi(x) = \frac{\delta}{\delta A(x)} \int L \, dx = S \epsilon \delta \frac{\partial A}{\partial t}(x)
\]
and we promote \(\hat{A}, \hat{\Pi}\) to the status of operators by imposing the canonical commutation relations
\[
[\hat{A}(x_1), \hat{\Pi}(x_2)] = i\hbar \delta(x_1 - x_2). \tag{2.2}
\]

Starting from the field operator \(\hat{A}(x)\) we can define the *destruction operator* associated with a particular solution of the Eq. \(2.1\) \(f(x)\) by taking the projection on it of the field operator
\[
\hat{a}(f) = (f, \hat{A}) \tag{2.3}
\]
and using the canonical commutation relations \(2.2\) we find
\[
[\hat{a}(f), \hat{a}^+(g)] = (f, g)
\]
\[
[\hat{a}(f), \hat{a}(g)] = -(f, g^*)
\]
\[
[\hat{a}^+(f), \hat{a}^+(g)] = -(f^*, g).
\]

From these relations it follows that we can obtain the usual commutation rules for creation and destruction operators if we are able to decompose the space of the complex solutions of the wave equation in a direct sum of a positive norm subspace and its complex conjugate \(S_p \oplus \overline{S_p}\), such that
\[
(f, f) > 0 \quad \forall f \in S_p
\]
and
\[
(f, g^*) = 0 \quad \forall f, g \in S_p.
\]

The first condition means that it is possible to rescale each \(f\) is such a way that the operators \(\hat{a}(f), \hat{a}^+(f)\) behave effectively as creation and destruction operators for an harmonic oscillator:
\[
[\hat{a}(f), \hat{a}^+(f)] = 1, \quad [\hat{a}(f), \hat{a}(f)] = 0, \quad [\hat{a}^+(f), \hat{a}^+(f)] = 0.
\]

The second condition tell us that creation and destruction operators commute between themselves.

We can now define the vacuum state \(|0>\) with the property
\[
\hat{a}(f)|0> = 0 \quad \forall f \in S_p.
\]

If \(\mu\) and \(\epsilon\) are time independent the wave equation \(2.1\) is time independent also, and admit as solutions *positive frequency* modes of the form
\[
f_k(x, t) = e^{-i\omega_k t} \tilde{f}_k(x) \tag{2.4}
\]
where \( \omega_k > 0 \). Note that \( f_k^*(x, t) \) is also a solution, which we can call a negative frequency mode. From the time invariance of our scalar product it follows that we can normalize these solutions in such a way that

\[
(f_k, f_{k'}) = \delta_{kk'}, \quad (f_k, f_k^*) = (f_k^*, f_{k'}) = 0.
\]

In this static case the Hamiltonian reduces to the sum of independent harmonic oscillators, one for each positive frequency mode. In fact we can expand the field operator

\[
\hat{A}(x, t) = \sum_k \left( \hat{a}_k \tilde{f}_k(x) e^{-i\omega t} + \hat{a}_k^+ \tilde{f}_k^*(x) e^{i\omega t} \right)
\]

(2.5)

and inserting in the Hamiltonian

\[
\hat{H} = \frac{\epsilon_0 S}{2} \int \left[ \epsilon \left( \frac{\partial \hat{A}}{\partial t} \right)^2 + \frac{c^2}{\mu} \left( \frac{\partial \hat{A}}{\partial x} \right)^2 \right] dx
\]

we obtain

\[
\hat{H} = \sum_k \hbar \omega_k \left( \hat{a}_k^+ \hat{a}_k + \frac{1}{2} \right).
\]

This expression tells us that the electromagnetic field can be described as an assembly of harmonic oscillators, one for each mode of the field. From Eq. (2.5) it is easy to recognize that the quantized nature of the field is completely contained in the creation and destruction operators, while the spatial dependence is encoded in the functions \( f_k \).

It is convenient to introduce the Hermitian quadrature operators \( \hat{p}_k \) and \( \hat{q}_k \), which are simply the “momentum” and the “coordinate” of the \( k \)-th oscillator, as

\[
\hat{a}_k = \frac{1}{\sqrt{2}} (\hat{q}_k + i \hat{p}_k), \quad \hat{q}_k = \frac{1}{\sqrt{2}} (\hat{a}_k^+ + \hat{a}_k), \quad \hat{p}_k = \frac{i}{\sqrt{2}} (\hat{a}_k^+ - \hat{a}_k)
\]

with commutation rules

\[
[\hat{q}_k, \hat{p}_{k'}] = i \delta_{kk'}.
\]

Note that we can always make the canonical transformation

\[
\hat{a}_k' = e^{i\theta_k} \hat{a}_k
\]

which preserve the commutation relations and the form of the Hamiltonian. Under this transformation the quadratures are rotated

\[
\hat{q}_k' = \hat{q}_k \cos \theta + \hat{p}_k \sin \theta
\]
\[
\hat{p}_k' = -\hat{q}_k \sin \theta + \hat{p}_k \cos \theta
\]

which means that we are free to redefine the phase reference of our mode. It will be useful to introduce a general quadrature operator in the direction \( \theta \) as

\[
\hat{q}_\theta = \hat{q} \cos \theta + \hat{p} \sin \theta.
\]

(2.6)

**Exercise 2** Find explicitly the positive frequency modes when there is a very thin barrier in \( x = 0 \) described by the transmission and reflection coefficients \( t_-, r_- \) (from left to right) and \( t_+, r_+ \) (from right to left).
2.1.2 Quasiprobability distributions

An useful tools to understand intuitively the peculiarities of a state of the electromagnetic field is the Wigner function, which is defined by

\[ W(p, q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx} \left( q - \frac{x}{2} \right) \left| \hat{\rho} \right|_x \left( q + \frac{x}{2} \right) dx \]  

(2.7)

where \( \hat{\rho} \) is the density operator of the system. It is mainly used when we are interested in study quantum correction to classical laws. The reason is that to some extent it resembles a probability distribution in the phase space, which in our case is the space of the quadratures. Of course it is not possible to define a phase space probability distribution in a strict sense for a quantum system because the two quadrature (coordinate and momentum) cannot be measured with arbitrary precision at the same time.

However it is possible to verify that if we can obtain the probability distribution for a generic quadrature integrating over the orthogonal one (see Eq. (2.6))

\[ P(q_\theta) = \int_{-\infty}^{\infty} W(q, p) dq \]

which means that \( W(p, q) \) gives the correct marginal distributions. Another important property give us a method to evaluate the expectation value of an observable. It is the overlap formula

\[ \text{Tr} \left( \hat{O}_1 \hat{O}_2 \right) = 2\pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_{\hat{O}_1}(p, q) W_{\hat{O}_2}(p, q) dp dq \]

that express the trace of the product of two operators as a superposition integral. The function \( W_\hat{O} \) is a generalization of (2.7) obtained with the substitution \( \hat{\rho} \rightarrow \hat{O} \). We can now evaluate expectation values as superposition integrals using

\[ < \hat{O} >= \text{Tr} \left( \hat{O} \hat{\rho} \right) = 2\pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(p, q) W_\hat{O}(p, q) dp dq . \]

It is interesting to derive the time evolution of the Wigner function. We can proceed in a direct way evaluating the time derivative of Eq. (2.7). In this way we obtain

\[ \frac{\partial W}{\partial t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx} \left( q - \frac{x}{2} \right) \left| \hat{H}, \hat{\rho} \right|_x \left( q + \frac{x}{2} \right) dx \]  

(2.8)

and choosing the Hamiltonian \( \hat{H} = \frac{\hat{p}^2}{2M} + U(q) \) we get a partial derivative equation

\[ \frac{\partial W(p, q, t)}{\partial t} = \left( -\frac{p}{M} \frac{\partial}{\partial q} + \sum_{l=0}^{\infty} \left( \frac{i}{2} \right)^{2l+1} \frac{1}{(2l+1)!} \frac{d^{2l+1}U(q)}{dq^{2l+1}} \frac{\partial^{2l+1}}{\partial p^{2l+1}} \right) W(p, q, t) \]

which determine the time evolution. If we rewrite it in the form

\[ \left( \frac{\partial}{\partial t} + \frac{p}{M} \frac{\partial}{\partial q} - \frac{dU(q)}{dq} \frac{\partial}{\partial p} \right) W(p, q, t) = \sum_{l=1}^{\infty} \left( \frac{i}{2} \right)^{2l} \frac{1}{(2l+1)!} \frac{d^{2l+1}U(q)}{dq^{2l+1}} \frac{\partial^{2l+1}}{\partial p^{2l+1}} W(p, q, t) \]  

(2.9)

we recognize on the left a classical Liouville equation. The terms on the right goes to zero when \( \hbar \rightarrow 0 \), but also when the potential is quadratic. This means that if the Hamiltonian is quadratic the Wigner function has a time evolution which is the same of a classical distribution functions. In the following we will give some examples of Wigner distributions for peculiar states.

**Exercise 3** Prove the equation 2.9.
2.1.3 Fock states

Fock states are the eigenstates of the photon number operators \( \hat{n}_k = \hat{a}_k^\dagger \hat{a}_k \). They can be constructed from the vacuum state with a repeated application of the mode creation operators. In fact if we apply a creation operator to an eigenstate of \( \hat{n} \) with eigenvalue \( n \) we obtain an eigenstate with eigenvalue \( n + 1 \)

\[
\hat{n} \hat{a}^\dagger |n> = \hat{a}^\dagger \hat{n} |n> = \hat{a}^\dagger \hat{a} |n> + \hat{a}[\hat{a}, \hat{a}^\dagger] |n> = (n+1)\hat{a}^\dagger |n>
\]

and in the same way with the application of a destruction operator we obtain a eigenstate with eigenvalue \( n - 1 \)

\[
\hat{n} \hat{a} |n> = \hat{a}^\dagger \hat{a} |n> = \hat{a} \hat{n} |n> + [\hat{a}^\dagger, \hat{a}] |n> = (n-1)\hat{a}^\dagger |n>.
\]

Imposing the correct normalization we find

\[
\hat{a} |n> = \frac{\sqrt{n}}{\sqrt{n!}} |1> \quad \hat{a}^\dagger |n> = \frac{\sqrt{n+1}}{\sqrt{(n+1)!}} |1>
\]

If \( n \) were not integer we could construct a state with an arbitrary negative eigenvalue, but this is not possible because the expectation value of \( \hat{n} \) is lower limited. In fact using the quadrature operators previously introduced we get

\[
<\hat{n}> = \frac{1}{2} <p^2> + \frac{1}{2} <q^2> - \frac{1}{2} \geq -\frac{1}{2}.
\]

This means that \( n \) is in fact an integer, and that

\[
\hat{n}|0> = \hat{a}^\dagger \hat{a}|0> = 0.
\]

Exercise 4 Show that if \( \hat{a}\phi_0(q) \neq 0 \) the solution of Eq. (2.10) is not physically acceptable.

The occupation number state is given explicitly by

\[
|n> = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}} |0>
\]

and its wave function can be obtained from this:

\[
\phi_n(q) = \frac{1}{\sqrt{n!}} \left[ \frac{1}{\sqrt{2}} \left( q - \frac{\partial}{\partial q} \right) \right] \phi_0(q) = \frac{H_n(q)}{\sqrt{2^n n! \sqrt{\pi}}} \exp \left( -\frac{q^2}{2} \right)
\]

where \( H_n \) are Hermite’s polynomials.

Exercise 5 Evaluate the Wigner function of a number state.
2.1.4 Coherent states

Coherent states are autostates of the destruction operator

$$\hat{a}|\alpha> = \alpha|\alpha> .$$

It is convenient to introduce the displacement unitary operator

$$\hat{D}(\alpha) = \exp \left( \alpha \hat{a}^+ - \alpha^* \hat{a} \right)$$

which is called in this way because it displace the destruction operator by the complex c-value \(\alpha\):

$$\hat{D}^+(\alpha) \hat{a} \hat{D}(\alpha) = \hat{a} + \alpha .$$

Now from

$$\hat{a} \hat{D}(-\alpha)|\alpha> = \hat{D}(-\alpha) \hat{D}^+(-\alpha) \hat{a} \hat{D}(-\alpha)|\alpha> = \hat{D}(-\alpha) (\hat{a} - \alpha)|\alpha> = 0$$

it follows that a coherent state can be written as

$$|\alpha> = \hat{D}(\alpha)|0> .$$

To evaluate the wave function we use the Baker-Hausdorff formula to rewrite the displacement operator in the alternate form

$$\hat{D}((q_0 + ip_0)/\sqrt{2}) = \exp \left( -i \frac{p_0 q_0}{2} \right) \exp (ip_0 \hat{q}) \exp (-i q_0 \hat{p})$$

which applied on the vacuum wave function (\(\exp(-i q_0 \hat{p})\) is a translation operator in the coordinate representation) gives

$$<q|\alpha> = \frac{1}{\pi^{1/4}} \exp \left( -\frac{(q - q_0)^2}{2} + ip_0 q - i \frac{p_0 q_0}{2} \right) . \hspace{1cm} (2.11)$$

It is instructive to project a coherent state in autostates of the occupation number

$$<n|\alpha>=<0|\frac{\hat{a}^n}{\sqrt{n!}}|\alpha> = \frac{\alpha^n}{\sqrt{n!}} <0| \exp \left( \alpha \hat{a}^+ - \alpha^* \hat{a} \right) |0> = \frac{\alpha^n}{\sqrt{n!}} e^{-\frac{1}{2} |\alpha|^2} .$$

We see that the probability distribution for the number of photons in a coherent state is a Poissonian with mean value \(|\alpha|^2\):

$$P_n = \frac{|\alpha|^{2n}}{n!} e^{-\frac{1}{2} |\alpha|^2} .$$

The Wigner function of a coherent state is obtained inserting the wave function inside Eq. (2.7) and it is, as could be expected, a Gaussian function

$$W_\alpha(q,p) = \frac{1}{\pi} \exp \left[ -(q - q_0)^2 - (p - p_0)^2 \right] \hspace{1cm} (2.12)$$

displaced in the point \((q_0, p_0)\) and with the same width in all the directions. This is an expression of the fact that the quantum fluctuations are equally distributed between the two quadratures. Note also that \(W(q,p)\) is always positive. This is a signature of the "semiclassical" nature of a coherent state.
2. Quantum optics

2.1.5 Minimum uncertainty states

In this section we want to give a complete characterization of a minimum uncertainty state. This problem was originally solved by Pauli using the method we describe now. First of all we can restrict to states with an expectation value for the destruction operator equal to zero. The reason is that the uncertainty of the state does not depend from it, because the two states 

\[ |\psi> = \hat{D}(\alpha)|\phi> \]

have the same quantum fluctuations. We start observing that if \( \phi(q) \) is the position wave function of our state we can construct a nonnegative quantity

\[ \delta = \left| \frac{q}{2\Delta^2 q} \phi + \frac{\partial \phi}{\partial q} \right|^2 \]

which expanded explicitly reads

\[ \delta = \frac{q^2 - 2\Delta^2 q}{4(\Delta^2 q)^2} \phi^* \phi + \frac{\partial \phi^*}{\partial q} \frac{\partial \phi}{\partial q} + \frac{1}{2} \frac{\partial}{\partial q} \left( \frac{q}{\Delta^2 q} \phi^* \phi \right) \geq 0. \]

If we integrate this expression throwing away the total differential we get the Heisenberg inequality. But this means that the minimum uncertainty is equivalent to \( \delta = 0 \), so in that case the wave function must be a normalized solution of

\[ \frac{q}{2\Delta^2 q} \phi + \frac{\partial \phi}{\partial q} = 0 \]

which is a Gaussian

\[ \phi(q) = \langle q | \phi > = \frac{1}{(2\pi \Delta^2 q)} \exp \left( -\frac{q^2}{4\Delta^2 q} \right) \]

but with a width parameter \( \Delta^2 q \) arbitrary. So a minimum uncertainty status has the same Gaussian wave function of a coherent state, but \( \Delta q \) can be different (larger or smaller) from \( \sqrt{\hbar / 2} \). How course the Heisenberg relation must be valid, and in fact if we evaluate the wave function in the momentum representation we get

\[ <p | \phi > = \left( \frac{4\Delta^2 q}{2\pi} \right) \exp \left( -\Delta^2 q p^2 \right). \]

2.1.6 Squeezed states

A squeezed state is generally speaking a state where the fluctuations of a quadrature operator is less than \( \sqrt{\hbar / 2} \). This can be achieved, but not only, with a minimum uncertainty state, in which case we know that the marginal distribution along two orthogonal quadrature must be a Gaussian, with

\[ \Delta q_\theta \Delta q_{\theta + \theta} = \frac{\hbar}{2}. \]

If we choose for definiteness the quadratures \( q \) and \( p \) we can parameterize the squeezing as a function of a parameter \( \zeta \) writing

\[ \Delta^2 q = \frac{\hbar}{2} e^{-2\zeta} \]
\[ \Delta^2 p = \frac{\hbar}{2} e^{+2\zeta} \]

and we can squeeze the vacuum wave function just by rescaling the coordinate keeping fixed the normalization

\[ \phi_\zeta(q) = e^{\zeta/2} \phi_0(e^{\zeta} q) \]

(2.13)

obtaining a Gaussian state.
Figure 2.1: Wigner’s function for states with different kinds of squeezing. From the top: coherent state, phase squeezed state, quadrature squeezed state, amplitude squeezed state. The origin of the quadrature plane is shown.
Exercise 6 Find the unitary operator which transform a wave function accordingly with Eq. 2.13.

As we see in the previous section a minimum uncertainty state is a displaced Gaussian states. It follows that all minimum uncertainty states can be written as displaced squeezed vacuums

$$|\alpha, \zeta> = \hat{D}(\alpha)\hat{S}(\zeta)|0>.$$  

The Wigner's function of a squeezed state is easily obtained transforming the coherent state's one. Inserting the squeezing operator we get

$$W_{\alpha,\zeta}(p,q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx} \left( e^{\zeta (q-x^2/2)} \hat{p} e^{\zeta (q+x^2/2)} \right) dx.$$  

If we change the integration variable and set $$\hat{\rho} = |\alpha><\alpha|$$ we recognize that the squeezed distribution is a "stretched" version of the coherent one:

$$W_{\alpha,\zeta}(p,q) = W_{\alpha}(e^{-\zeta}p, e^{\zeta}q).$$

Some particular cases are showed in Figure 2.1 on page 17, where moving for the top it is depicted the Wigner's function of

1. a coherent state, which is the displaced Gaussian of Eq. 2.12.
2. a “phase squeezed” state, which is a squeezed state with $$\zeta > 0$$. The name is motivated from the fact that the fluctuations of the mode’s phase are reduced.
3. a “quadrature squeezed” state. Here the quadrature with the largest reduction of fluctuations is neither $$q$$ nor $$p$$, but a generic one $$q_0$$. This state can be obtained taking applying a rotation after the squeezing. The combined action of the squeeze and rotation operator can be obtained defining a complex squeezed parameter

$$\zeta = r_s e^{2i\theta_s}$$

and rewriting

$$\hat{S}(\zeta) = \hat{S}(\zeta) = \exp \left[ \frac{1}{2} (\zeta^* a\hat{a} - \zeta a^+ \hat{a}^+) \right].$$

The parameter $$r_s$$ represent the degree of squeezing and $$\theta_s$$ is the angle of the squeezing direction.
4. an “amplitude squeezed” state. In this case the fluctuation of the field amplitude are reduced.

The variance for an arbitrary quadrature can be written as

$$\Delta q_\theta^2 = \cosh 2r_s - \sinh 2r_s \cos 2(\theta - \theta_s)$$ (2.14)

which in the quadrature plane is an ellipse that can be drawn to describe synthetically the state, as we will do in the last part of these notes.

Exercise 7 Find the eccentricity and the inclination of the ellipse described by Eq. 2.14.
2.2 Quantum networks

In order to detect a gravitational wave we typically monitor the relative position of a set of reference masses using the light generated by a laser. The laser beam, that we will approximate with a coherent state of a well defined mode of the electromagnetic field, is transmitted, reflected and recombined by the different element of the optical apparatus.

In this section we will study the more important properties of the interaction between the quantized modes of the electromagnetic field and the more common optical elements. We will work always linearizing in the fluctuations, considering these as very small quantities. This is usually a legitimate procedure and the theory can be simplified a lot.

Before starting a detailed analysis of some particular optical device we discuss some results for a generic one which act on a set of modes in a linear way. As a first step we introduce the notion of output and input modes.

- An input mode is a solution of the wave equation for the electromagnetic field in presence of the device whose ingoing part reduces to a single beam. In other word, an input mode represent a single beam which interact with the device and is scattered by it in a superposition of beams. We will indicate the spatial part of the $i$-th input mode with $\tilde{f}_i(x)$.

- An output mode is a solution of the wave equation for the electromagnetic field in presence of the device whose outcoming part reduces to a single beam. This can be seen usually as a linear superposition of input modes: the spatial part of the $j$-th output mode will be indicated by $\tilde{g}_j(x)$.

The electromagnetic field can be written as a combination of input modes

$$\hat{A}(x,t) = \sum_{i=1}^{N_{IN}} \left( \hat{a}_i \tilde{f}_i(x) e^{-i\omega t} + \hat{a}^+_i \tilde{f}^*_i(x) e^{i\omega t} \right)$$

which we parameterized as usual with the operators $\hat{a}_i$. Using the property 2.3 we obtain the relation between these and the operators $\hat{b}_i$ which describe the output modes.

$$\hat{b}_i = <\tilde{g}_i, \hat{A}> = \sum_{j=1}^{N_{IN}} \left( \hat{a}_j <\tilde{g}_i, \tilde{f}_j > + \hat{a}^+_j <\tilde{g}_i, \tilde{f}^*_j > \right)$$

which can be written concisely as

$$\bar{b} = \mathcal{G}\bar{a}$$

where we introduced the condensed notation

$$\bar{a}^T = (\hat{a}_1, \ldots, \hat{a}_{N_{IN}}, \hat{a}^+_1, \ldots, \hat{a}^+_{N_{IN}})$$

and

$$\bar{b}^T = (\hat{b}_1, \ldots, \hat{b}_{N_{OUT}}, \hat{b}^+_1, \ldots, \hat{b}^+_{N_{OUT}}).$$

We want both input and output modes to have the correct commutators, which means

$$\mathcal{G}^{(N_{OUT})} \left[ \bar{b}, \bar{b}^+ \right] = \mathcal{G} \left[ \bar{a}, \bar{a}^+ \right] \mathcal{G}^+ = \mathcal{G}^{(N_{IN})} \mathcal{G}^+$$

(2.15)

where the matrix $\mathcal{G}^{(k)}$ is defined by

$$\mathcal{G}^{(k)} = \begin{pmatrix} I_k & 0 \\ 0 & -I_k \end{pmatrix}.$$
The relation (2.15) imply that $N_{IN} = N_{OUT}$, and that the matrix $\mathcal{S}$, which is called scattearing matrix, is square and quasi-unitary:

$$G \mathcal{S}^\dagger G = \mathcal{S}^{-1}.$$ 

This means that in the general case the number of input and output modes must be the same, and that there are some constraints on the values of the scattering matrix elements.

### 2.2.1 Beam splitter

A basic building block for an optical apparatus is a beam splitter. This is in the ideal case a reversible and lossless device in which two incidents beams interfere to produce two output beams. It is represented schematically in Figure 2.2 on page 20, where the two input port and the two output ones are indicated.

![Figure 2.2: Schematization of an ideal beam splitter, and its diagrammatical representation as a linear relation between two input and two output modes.](image)

This object can be realized, for example, with a slab of dielectric material and their classical properties can be evaluated. In a simplified model the first input beam is partly reflected on the second output, and partly transmitted to the first. In the same way the second input beam is partly reflected (on the first output) and partly transmitted (on the second one). As this is a passive element, no new excitation are created and this means that the total photon number operator must be of the same form when written in terms of the $\hat{a}$ and $\hat{b}$ operators. In other words

$$\hat{a}^+ \cdot \hat{a} = \hat{b}^+ \cdot \hat{b} = \hat{a}^+ \cdot \mathcal{S}^\dagger \mathcal{S} \cdot \hat{a},$$

and this means that $\mathcal{S}$ must commute with $G$, which is possible only if it is of the form

$$\mathcal{S} = \begin{pmatrix} S & 0 \\ 0 & S^* \end{pmatrix}$$

where $S$ is unitary. In simpler words, the scattering array does not mix creation and destruction operators.

The reduced scattering matrix $S$ gives a complete description of the behavior of our system. A general parameterization of a $2 \times 2$ unitary matrix can be given in term of four angular parameters:

$$S = e^{i\frac{\pi}{2}\Lambda} \begin{pmatrix} e^{i\frac{\pi}{2}\psi} & 0 \\ 0 & e^{-i\frac{\pi}{2}\psi} \end{pmatrix} \begin{pmatrix} \cos \frac{1}{2}\theta & \sin \frac{1}{2}\theta \\ -\sin \frac{1}{2}\theta & \cos \frac{1}{2}\theta \end{pmatrix} \begin{pmatrix} e^{i\frac{\pi}{2}\phi} & 0 \\ 0 & e^{-i\frac{\pi}{2}\phi} \end{pmatrix}.$$
Usually the phases can be adsorbed in a redefinition of the input and output states, so the really important piece is the rotation which mix together input and output states. With this simplification it is usual to reparameterize $S$ as

$$S = \begin{pmatrix} t & -r \\ r & t \end{pmatrix},$$

where the transmission and reflection parameters $r$, $t$ must satisfy the condition

$$t^2 + r^2 = 1$$

which is simply energy conservation.

Figure 2.3: Input and output modes of a beam splitter. From left to right: first and second input modes, first and second output modes. The arrows indicate that a beam is present.

Note that our beam splitter model is good also for a semitransparent mirror, which act in exactly the same way on the beams.

### 2.2.2 The role of the losses

A real optical device is never lossless. Incoming field can be absorbed or scattered away by defects, and we need a model to describe in the correct way this kind of effects. Let us consider a simple optical device with a single input and a single output. The relation between destruction operators for the input and output modes is just a number

$$\hat{b}_1 = S \hat{a}_1$$

and since $\hat{a}_1$ and $\hat{b}_1$ must have both the usual commutation rules $S$ can be only a phase. So there is apparently no space for losses in a quantized framework.

The key to solve this problem reveals also a very important aspect of the losses. If there is a coupling between the electromagnetic mode and other (unknown) degrees of freedom $X_i$ which generates dissipation, the same coupling must be able to couple the fluctuations of these $X_i$'s to the electromagnetic mode. This means that our single-input, single-output device with losses could be modeled by a lossless two-input, two-output device. The losses can described in this framework as the fraction of the field in the first input which is reflected on the second output in Figure 2.2 on page 20, and the vacuum field incoming on the second input is reflected in part on the first output, adding its fluctuations to the ones of the beam entering in the first input port.

In other words, the unitarity of the quantum theory tell us that a loss is always connected to a coupling to vacuum fluctuations, and prescribe that these fluctuations are exactly proportional to the losses.

---

1 Many other conventions are used in literature, which correspond to different choices of the phases $\phi$ and $\psi$. Starting for a real physical model of the beam splitter it is of course possible to find out what are the real phases involved, which will depend on the particular case.
Figure 2.4: A simple model of photodiode with quantum efficiency \( \eta \). Only a fraction \( \eta \) of the incoming energy is converted in photoelectrons, and this is equivalent to an amplitude loss \( \sqrt{\eta} \) in the input beam. The beam incoming on the detector get an additional contribution of fluctuations.

Let us see this in a specific example, that will be very important in the following. Suppose that we have an incoming squeezed state \( |\alpha, \zeta\rangle \) which propagates for a while inside a material with losses, so that the outcoming field intensity is reduced by a factor \( \eta \). The equivalent beam splitter model can be written as

\[
\begin{pmatrix}
\hat{b} \\
\hat{b}_v
\end{pmatrix} = \begin{pmatrix}
\sqrt{\eta} & -\sqrt{1-\eta} \\
\sqrt{1-\eta} & \sqrt{\eta}
\end{pmatrix} \begin{pmatrix}
\hat{a} \\
\hat{a}_v
\end{pmatrix}
\] (2.16)

and this gives immediately a relation for the fluctuation of a quadrature of the output beam

\[
\Delta q_{\theta_s}^2 = \eta [\cosh 2r_s - \sinh 2r_s \cos 2(\theta - \theta_s)] + (1 - \eta) .
\] (2.17)

We obtain for the minimum uncertainty

\[
\Delta q_{\theta_s}^2 = \eta e^{-2r_s} + (1 - \eta)
\]

which tell us that the presence of losses destroy very rapidly the squeezing property of a state. It is instructive to note that it is possible to relate the Wigner’s distributions of the input and output field. The result is given by

\[
W_{out}(p, q) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W_{in}(\sqrt{\eta}q - \sqrt{1-\eta}q', \sqrt{\eta}p - \sqrt{1-\eta}p') \\
\times W_v(\sqrt{1-\eta}q + \sqrt{\eta}q', \sqrt{1-\eta}p + \sqrt{\eta}p')dq'dp'
\]

where \( W_v \) is the Wigner’s function of the vacuum. This relation can be understood qualitatively in the following way: first the variables \( p, q \) are rescaled, then the input’s Wigner’s function is convoluted with the Wigner’s function of the vacuum. This convolution introduce a “smoothing” which tends to destroy the squeezing.

As a concrete example we can model a photodiode with a given quantum efficiency as an ideal one with a beam splitter inserted before, as in Figure 2.4.

2.2.3 Moving mirror: quadrature fields and radiation pressure.

Another building block for the description of the optical apparatus we are interested to is a moving mirror interacting with a plane wave. We will use this simple example
to introduce the useful formalism of quadrature fields and to evaluate the effect of the radiation pressure.

We suppose the mirror's motion composed by small oscillations around an equilibrium point which we choose as the origin. For a mirror with reflectivity $r$ we can write the reflected field as a function of the incoming one as

$$A_R(0, t) = r A_I(0, t - \frac{2}{c} \delta X(\tau))$$

where the delay time $\tau$ is defined by $\delta X(\tau) = c(t - \tau)$ and we suppose the incoming field moving in the increasing coordinate direction. The physical picture is that the reflected field is simply the incident one, delayed by the time it takes to move from $X_0$ to the mirror's position and back. We can linearize the previous expression in the small displacements of the mirror, obtaining

$$A_R(0, t) = r A_I(0, t) - r \frac{\partial A_I}{\partial t}(0, t) \frac{2}{c} \delta X(\tau)$$

where we recognize that up to this order we can set $\tau = t$. Transforming in the frequency domain we get

$$\tilde{A}_R(0, \omega) = r \tilde{A}_I(0, \omega) - \int d\omega' \frac{2i\omega'}{c} \tilde{A}_I(0, \omega') \delta \tilde{X}(\omega - \omega'). \tag{2.18}$$

In the linear approximation each frequency of mirror’s displacement act independently. Suppose for example

$$\delta \tilde{X}(\omega) = \delta X \delta(\omega - \Omega) + \delta X^* \delta(\omega + \Omega)$$

then we get

$$\tilde{A}_R(0, \omega) = r \tilde{A}_I(0, \omega) - \frac{2i\omega}{c} \left[ \frac{\omega - \Omega}{\omega} \delta X \tilde{A}_I(0, \omega - \Omega) + \frac{\omega + \Omega}{\omega} \delta X^* \tilde{A}_I(0, \omega + \Omega) \right]$$

which shows that the mirror motion at the frequency $\Omega$ generate a couple of sidebands. In the typical situation we are interested to the incoming field is a coherent state of amplitude $\alpha_I$ at a given carrier frequency $\omega_C$, which models the laser’s beam. This means that we can write the incoming field operator as

$$\tilde{A}_I(x, \omega) = \alpha_I(x) \delta(\omega - \omega_C) + \alpha_I^*(x) \delta(\omega + \omega_C) + \delta \hat{A}_I(x, \omega)$$

where $\delta \hat{A}_I(x, \omega)$ describes the vacuum fluctuations, so that $\forall \omega$ it is $< \delta \hat{A}_I(x, \omega) >= 0$. If we substitute inside Eq. (2.18) and we separate the zero and first order in the fluctuations we get the relations

$$\alpha_R(0) = \alpha_I(0)$$

$$\delta \hat{A}_R(0, \omega) = r \delta \hat{A}_I(0, \omega) - \frac{2i\omega c}{\omega} \alpha_I(0) \delta X(\omega - \omega_C) - \frac{2i\omega c}{\omega} \alpha_I^*(0) \delta X(\omega + \omega_C) \tag{2.19}$$

In the applications we are interested to the argument of the Fourier transform of the mirror’s displacement will be very small compared with the frequency of the carrier, typically lower than few tenth of kHz. In the following we indicate it with the capital letter $\Omega$.

### 2.2.3.1 Quadrature fields

It is convenient to introduce a new class of operators, which describes the peculiar excitations that are created by the interaction between the carrier and the mirror. Looking at the previous expression we are tempted to define the quadrature fields as

$$\hat{A}^{(\theta)}(x, \Omega) = \frac{1}{\sqrt{2}} \left( e^{i\theta} \hat{A}(x, -\omega_C + \Omega) + e^{-i\theta} \hat{A}(x, \omega_C + \Omega) \right). \tag{2.20}$$
If we transform to the time domain we recognize that \( \hat{A}(\theta) \) is the modulation of the component of carrier’s field which rotate with phase \( \theta \). This is the basis of a convenient formalism to analyze the effects we are interested to, introduced by Caves and Schumaker \[?, ?\]. The main idea is that the fundamental excitations of the beam we are interested to study can be described as a couple of photons coherently excited in two symmetric sidebands of the carrier field. As we will see in a moment this is for example the kind of excitation created or destroyed in the linearized interaction with a moving mirror.

We can see that the action of a moving mirror on these operators is somewhat simpler than Eq. (2.19). We can write

\[
\delta \hat{A}_R^{(\theta)}(\Omega) = r \delta \hat{A}_R^{(\theta)}(\Omega) - \frac{2i\omega c r}{c} \left\{ \alpha_I(0) \left[ e^{i\theta} \delta X(\omega c + \Omega) + e^{-i\theta} \delta X(\Omega) \right] + \alpha_I^*(0) \left[ e^{i\theta} \delta X(\Omega) + e^{-i\theta} \delta X(2\omega c + \Omega) \right] \right\}
\]

but we expect that the mirror motion will give a negligible contribution at optical frequencies, so we can further simplify the previous expression to

\[
\delta \hat{A}_R^{(\theta)}(\Omega) = r \delta \hat{A}_R^{(\theta)}(\Omega) - \frac{2i\omega c r}{c} \left[ \alpha_I(0) e^{-i\theta} + \alpha_I^*(0) e^{i\theta} \right] \delta X(\Omega)
\]

There are two independent components of a quadrature field, that we can choose for example as \( \hat{A}(0) \) and \( \hat{A}(\pi/2) \). If we introduce the vectorial notation

\[
\hat{A}(\Omega) = \left( \begin{array}{c} \hat{A}(0) \\ \hat{A}(\pi/2) \end{array} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \hat{A}(\omega c + \Omega) + \hat{A}(\omega c - \Omega) \\ -i\hat{A}(\omega c + \Omega) + i\hat{A}(\omega c - \Omega) \end{array} \right)
\]

we can rewrite the reflection rule as

\[
\delta \hat{A}_R(\Omega) = \delta \hat{A}_I(\Omega) - \frac{2\omega c r}{c} \delta X(\Omega) \Sigma \Omega_I
\]

where \( \Sigma \) is a matrix of a \( \pi/2 \) rotation and \( \Omega_I \) is defined consistently with Eq. (2.21) as

\[
\Omega_I = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \alpha_I + \alpha_I^* \\ -i\alpha_I + i\alpha_I^* \end{array} \right).
\]

It is possible to give an intuitive interpretation of the introduced formalism. The classical amplitude of a coherent state can be represented drawing the vector \( \alpha \) in a plane. The fluctuations of the field are represented by the vector \( \delta \alpha \), and can be represented graphically with a “confusion area” with elliptical shape. This resembles a description given in terms of a quasiprobability distribution, however we must remember that

- a quasiprobability distribution can be used as a probability only in a limited sense
- the definition of quadrature operators given here involves combinations of field operators at different frequencies

In spite of that the picture we get is useful to have an intuitive feeling of what is happening. For example the reflection rule (2.22) tell us that the motion of the mirror introduce an additional fluctuation in the quadrature orthogonal (owing to the \( \Sigma \) matrix) to the classical amplitude. In the linearized approximation this can be interpreted as a phase fluctuation of the field (see Figure 2.5).

The usual transfer function formalism is easily generalized in this new representation: the simple rule is that a linear relation between two destruction operators like \( b(\omega) = T(\omega) a(\omega) \) becomes a linear relation between the corresponding quadrature vectors

\[
b(\Omega) = T_{2\Omega}(\Omega)
\]

where

\[
T = \frac{1}{2} \left( \begin{array}{ccc} T_+ & T_x & iT_+ - iT_x \\ -iT_+ + iT_x & T_+ + T_x \end{array} \right)
\]
Figure 2.5: The representation of a quadrature field for a coherent state. The classical field is represented by the displacement of the center of fluctuations from the origin. Fluctuations parallel to the classical field can be interpreted as amplitude fluctuations, orthogonal ones as phase fluctuations.

We used here the convention $T_x = T(\omega_0 \pm \Omega)$. Note that $T$ is a rotation when $T$ is frequency independent.

**Exercise 8** Derive explicitly the matrix $T(\omega_0 \pm \Omega)$.

A simple example is the free propagation of a field, which is described by a phase factor:

$$\hat{a}(\omega) \rightarrow e^{i\omega L} \hat{a}(\omega).$$

In the quadrature formalism this becomes

$$\bar{a}(\Omega) \rightarrow D(\Omega) \bar{a}(\Omega) = \left( \begin{array}{cc} \cos \frac{\omega L}{c} & -\sin \frac{\omega L}{c} \\ \sin \frac{\omega L}{c} & \cos \frac{\omega L}{c} \end{array} \right) e^{i\Omega / c} \bar{a}(\Omega),$$

which means that the quadrature vector is rotated and acquire a phase factor.

### 2.2.3.2 Radiation pressure

Suppose now that the mirror moves because it react to the radiation pressure of the beam. This give us a connection between $\delta \hat{X}$ and the quantum fluctuation of the beam. In our unidimensional model the radiation pressure force is simply the purely spatial component of the energy-momentum tensor, which is equal to the energy density

$$F_{RP}(x,t) = T_{xx}(x,t) = T_{tt}(x,t) = \frac{S_{\epsilon_0}}{2} \left[ \left( \frac{\partial A}{\partial t} \right)^2 + c^2 \left( \frac{\partial A}{\partial x} \right)^2 \right]$$
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where we set \( \epsilon, \mu = 0 \) because we are interested to apply this expression just before the mirror. If we evaluate the flux of this quantity on a surface located just before the mirror in the frequency domain we obtain an expression for the pressure whose fluctuating part can be written in the quadrature formalism as

\[
\delta F_{RP} = \frac{2\omega_0}{c} r^2 (\alpha_L^T \cdot \delta A_L - \alpha_R^T \cdot \delta A_R)
\]

where \( \alpha_{L,R} \) are the classical amplitudes of the fields impinging on the mirror from the left and right side, while \( \delta A_{L,R} \) are their fluctuating parts.

2.2.4 Suspended mirror

We can give now a complete characterization of the interaction with a suspended mirror, depicted schematically in Figure 2.6.

![Figure 2.6: Schematic view of a moving mirror. The mirror is displaced with respect to its reference positions, which is on the dashed line.](image)

The radiation pressure is connected to the mirror’s displacement by the motion’s equation. This one can be written in term of a mechanical susceptibility function \( \chi \). If in the frequency band of interest the mirror can be considered a free mass we get

\[
\delta x(\Omega) = \chi(\Omega) \delta F_{RP}(\Omega) + \delta x’(\Omega) \simeq -\frac{1}{M\Omega^2} \delta F_{RP}(\Omega) + \delta x’(\Omega)
\]

(2.24)

where we introduced \( \delta x’ \) to indicate displacements unrelated to the radiation pressure which could represent for example thermal noise effects or gravitational strain. Putting all together we get a complete characterization of the mirror as

\[
\begin{align*}
\delta b_1 &= t \delta u_1 + r \delta a_1 - 2k_0 r [\delta x’ + 2\chi k_0 r^2 (\alpha_a^T \delta u_1 - \alpha_u^T \delta a_1)] \Sigma \alpha_a \\
\delta u_1 &= -r \delta u_1 + t \delta a_1 - 2r k_0 [\delta x’ + 2\chi k_0 r^2 (\alpha_a^T \delta u_1 - \alpha_u^T \delta a_1)] \Sigma \alpha_u.
\end{align*}
\]

(2.25)

To simplify the notation we set \( \alpha_u = 0 \) and \( r = 1 \) (no beam incoming from the right, perfectly reflecting mirror) and we study the noise of the reflected field whose fluctuations are described by

\[
\delta b_1 = \delta a_1 - 2k_0 [\delta x’ + 2\chi k_0 \alpha_a^T \cdot \delta a_1] \Sigma \alpha_a.
\]

The relevant quantity that must be computed is the cross correlation array \( <\delta b_{1\nu} \delta b_{1\mu}^\dagger>_{\text{symm}} \) (here \( \text{symm} \) stands for operator symmetrization). Now the output vector \( \delta b_1 \) can be written explicitly as

\[
\delta b_1 = \begin{pmatrix}
\delta a(0)(\Omega) \\
\delta a(\pi/2)(\Omega)
\end{pmatrix} - \begin{pmatrix}
0 \\
4k_0^2 \chi \alpha_a^2 \delta a(0)(\Omega)
\end{pmatrix} - \begin{pmatrix}
0 \\
2k_0 r \alpha_1 \delta x’
\end{pmatrix}
\]
where the last term describes the effect of mirror's fluctuations of non quantum origin. Evaluating the covariance matrix of the first two pieces we get

\[
\langle \delta b_\Omega \delta b_\Omega^\dagger \rangle_{\text{symm}} = \begin{pmatrix}
1 & -K \\
-K & 1 + K^2
\end{pmatrix}
\]

where \( K = 4\chi k_0^2 \alpha_0^2 \) is connected to the response of the mirror to radiation pressure. The covariance matrix represent a distribution of Gaussian fluctuations which can be represented in the quadrature plane as a rotated ellipse. This means that the reflected beam cannot be represented anymore by a coherent state, but instead by a squeezed state. The squeezing induced by the reaction of the mirror to the radiation pressure is named ponderomotive squeezing.

**Exercise 9** Determine the shape of the ellipse which represent the covariance array 2.26

### 2.2.5 Resonant cavity

Another useful example is the evaluation of the state of the light reflected by a resonant cavity. To keep the notation simple we set \( r \approx 1 \) and we write the junction conditions for the fluctuating part of the fields

\[
\begin{align*}
\delta b &= -r\delta a + i\delta d - r\xi\delta X \alpha_a \\
\delta c &= t\delta a + r\delta d - r\xi\delta X \alpha_d \\
\delta d &= \delta c + \xi\delta Y \alpha_c \\
\end{align*}
\]

while for the classical amplitudes we get

\[
\begin{align*}
\alpha_b &= \alpha_a \\
\alpha_c &= \alpha_d = \frac{t}{1 - r}\alpha_a .
\end{align*}
\]

Solving we get

\[
\begin{align*}
\delta b &= \delta a + \frac{2}{1 - r}\xi L \Sigma \alpha_a \\
\delta L &= \frac{4\chi \xi}{1 - r}\alpha^T \cdot \delta a + \delta L'
\end{align*}
\]

where we neglected the modulation induced by the direct reflection on the front mirror, which is suppressed by a factor \( 1 - r \). The second equation gives the response of the cavity to radiation pressure, so that \( \chi = -(\mu \Omega^2)^{-1} \) where \( \mu \) is the reduced mass of the two mirrors. The final result is very similar to the one discussed in the previous subsection: the output state is a squeezed one.

Note that for a resonant cavity radiation pressure does not depend up to the first order from \( \delta L \). This is not true for a detuned cavity, where the linear dependence of radiation pressure from \( \delta L \) give rise to a stiffness (optical spring).

### 2.2.6 Michelson interferometer: optical noise

We discuss only a couple of aspects of this case, very schematically. The optical scheme is depicted in Figure 2.8 where the field \( a \) represent the injected laser beam.
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Figure 2.7: A resonant, totally reflecting cavity.

Figure 2.8: Schematic diagram of a VIRGO-like interferometer. A laser beam is injected from the left (field $a$).
The output field $d$ is the one which is measured, and is given by the superposition of the states which traversed the two cavities,

$$d = tC_x(-ra + tb) + rC_y(ta + rb) = tr(C_y - C_x)a + (t^2C_x + r^2C_y)b$$

where $t,r$ is the transmittivity and reflectivity of the beam splitter, while $C_{x,y}$ is an operator which represents the action of the horizontal and vertical cavity. If we suppose that the two cavity have the same parameters we see that the only piece proportional to the input field which can survive is proportional to a fluctuation of non-optical origin (a gravitational wave, or a noise which acts in different way on the two cavities). This means that the optical fluctuations enters in the system only from the dark port (the field $b$, which is typically in a vacuum state).

If there is an asymmetry between the two cavities this is no more true, and a fraction of quantum fluctuations of the laser beam will enter into the system.
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In an interferometric detector of gravitational waves like Virgo or LIGO the space-time metric’s perturbation is monitored comparing the relative positions of some test masses. The basic optical scheme is a Michelson’s interferometer, and in the linearized regime the phase shift is proportional to a position operator $\hat{Q}$ for an effective free mass $\mu$. As discussed previously a non zero commutator

$$\left[ \hat{Q}(t_1), \hat{Q}(t_2) \right] = \frac{i\hbar}{\mu}(t_2 - t_1)$$

enforces an indetermination relation which forbids repeated measurements of $\hat{Q}$ of infinite precision. Though there are some delicate points in the interpretation of this result [?] the physical picture is very simple: a position measurement introduces an indeterminacy of the momentum, which in turn acts back on the position during the following time evolution. A rigorous and complete discussion of the accuracy achievable with continuous measurements can be found for example in [?].

A general strategy to analyze the quantum noise of a gravitational wave interferometric detector consist in expressing output quadrature vectors as a function of input ones. At that point we can evaluate the output covariance array connected with the noise amplitude. In a Virgo-like interferometer we have for example

$$\hat{b} = e^{2i\beta} \begin{pmatrix} 1 & 0 \\ -K & 1 \end{pmatrix} \hat{a} + e^{i\beta} \sqrt{\frac{2K}{S_h^{\text{SQL}}}} \begin{pmatrix} 0 \\ h \end{pmatrix}$$

(3.1)

where $\beta = \arctan \Omega/\gamma$ is the phase delay cumulated by the sideband inside the cavities,

$$S_h^{\text{SQL}} = \frac{8h}{m\Omega^2 L^2}$$

and

$$K = \frac{I_0}{I_{\text{SQL}}} \frac{2\gamma^4}{\Omega^2 (\gamma^2 + \Omega^2)}$$

can be interpreted as a measure of the strength of the back action effects. $I_0$ is the true intensity of the laser and

$$I_{\text{SQL}} = \frac{mL^2\gamma^4}{4\omega_0}$$

is the laser’s intensity required to reach the optimal sensitivity at $\Omega = \gamma \equiv c/(LF)$, where $L$ is the length of the cavities, $F$ their finesse, and $m$ the mass of the mirrors.

In the usual detection scheme we measure the $\theta = \pi/2$ output quadrature, and from Eq. (3.1) we see that this give us an estimator of the gravitational strain which can be written as

$$\hat{h} = h + e^{i\beta} \sqrt{\frac{S_h^{\text{SQL}}}{2K}} (\hat{a}_{x/2} - K\hat{a}_0)$$
and from it we get the power spectrum of the optical noise which can be written as

\[ S_h = \frac{1}{2} S_{h}^{SQL} \left( \frac{1}{\mathcal{K}} + \mathcal{K} \right). \]  

(3.2)

The first term inside the braces in Eq. (3.2), which comes from the \( \hat{a}_{\pi/2} \) operator and decreases with an increasing laser’s power, can be interpreted as the shot noise contribution. The second term, which comes from \( \hat{a}_0 \) and increases with an increasing laser’s power, can be associated to radiation pressure. The proportionality of radiation pressure noise to \( \Omega^{-2} \) in the low frequency region is connected to the mechanical response of the mirrors, which are supposed here to behave as free bodies of mass \( m \).

The best compromise between shot and radiation pressure noise is reached at a particular frequency \( f_{SQL} \) which depends on the laser’s power, and the locus of all these minima of the strain effective noise amplitude is called \textit{standard quantum limit} (SQL), and it is given simply by \( \sqrt{S_{h}^{SQL}} \).

Looking at the sensitivity curves of current interferometers (for instance in Figure 3.1) we see that a reduction of thermal noise of at least two orders of magnitude is mandatory before the standard quantum limit will become an issue.

In spite of that in these years there has been a considerable activity around this subject. Many proposals for the evasion of standard quantum limit exist now, and in the following we will give a very concise overview of some of these.


3.1 Proposals for SQL evasion

Before starting the discussion one may wonder if standard quantum limit is a fundamental limit, as one could suspect considering that it is strictly connected to a fundamental principle of quantum mechanics. The answer is negative, because in detecting gravitational waves we are not really trying to measure the position of the test masses, but instead to get informations about a (classical) force which acts on them. For a detailed analysis of this point see [?].

We list now some current proposals for evading SQL. As there is no place for a complete discussion we point to the literature for the details. Also, we do not pretend to achieve completeness, and the list is based mainly on personal taste and knowledge.

3.1.1 Modification of input and/or output

The basic starting point of this kind of approach is the observation that the interaction of the laser beam with the free masses of the mirror generates ponderomotive squeezing. Put in simple terms, this means that as the mirror recoils in response to a fluctuation of the laser’s amplitude, a fluctuation of the beam’s phase is generated. Phase and amplitude quantum fluctuations are no more uncorrelated, as it is the case for a coherent state, and there is an appropriate quadrature of the relevant mode of the electromagnetic field which sees reduced quantum fluctuations. Using the relations— between

---

Figure 3.2: Signal to noise ratio in unities of $\frac{\hbar}{\sqrt{S_h}}$, for different values of the parameter $K = 0, 1/2, 1, 2, 10$ as a function of the measured quadrature angle.
quadrature operators of input and output modes we can calculate the covariance array for output quadratures, which can be written as

\[ S_{ij} = \frac{1}{2} \begin{pmatrix} 1 & -K \\ -K & 1 + K^2 \end{pmatrix} \]  \hspace{1cm} (3.3)

where the out of diagonal element different from zero is a clear signature of the correlation introduced between amplitude and phase fluctuations.

If we measure the quadrature at the angle \( \theta \) we get a squared signal to noise ratio given by

\[ \text{SNR}(\theta)^2 = \left( \frac{\hbar^2}{S_h} \right) \frac{(1 + K^2) \sin^2 \theta}{1 + K^2 \sin^2 \theta - 2K \sin \theta \cos \theta} \]  \hspace{1cm} (3.4)

where we used the fact that in a conventional interferometer the information about the signal is aligned with the \( \theta = \pi/2 \) quadrature.

We can exploit this result measuring at the output the quadrature with the best SNR ratio. In Figure 3.2 we plotted the square root of Eq. (3.4) in units \( \hbar/\sqrt{S_h} \) for different values of the parameter \( K \). In the usual case the quadrature \( \pi/2 \) is measured, and it is apparent that no improvement is obtained by increasing \( K \). It is also apparent that choosing an optimized quadrature we obtain an improvement of the sensitivity proportional to the level of ponderomotive squeezing. Note that, as \( K \) is a frequency dependent parameter, the optimal quadrature will depend on frequency also and appropriate measurement strategies must be devised [?].

Another observation is that as the squeezing parameter \( K \) becomes large the SNR gain becomes a very sensitive function of the quadrature angle.

Another possibility is the injection of a squeezed state in the dark port of the interferometer (the only one which contribute to output quantum noise). To see the effect of this option we simply reevaluate the covariance array (3.3) using a squeezed state as input state in Eq. (3.1).

By choosing an appropriate (frequency dependent) squeezing angle we find a quantum noise amplitude spectrum given by

\[ S_h = \frac{1}{2} S_{hQL} \left( \frac{1}{K} + K \right) e^{-2r} \]

which is reduced from the expression in Eq. (3.2) exponentially in the squeezing parameter \( r \).

All these possibilities and their combinations are discussed in detail in [?]: it is possible to completely suppress radiation pressure noise, which means that there are no obstructions in principle to reduce shot noise also by increasing laser power. The improvement obtainable injecting squeezed light depends on the degree of squeezing available.

### 3.1.2 Measurement of quantum non demolition observables

The two times commutator between the momentum operators for a free mass is zero, so there is no indetermination limit to the precision obtainable in a repeated measurement of the mirror’s velocity. The velocity of a free mass is an example of a quantum non demolition observable: more generally a conserved quantity is QND also (see [?] for a general discussion).

There are alternative optical schemes which give, at least approximately, a measurement connected with the speed of the mirrors: these will not have a standard quantum noise limit [? , ?].

As an example we can look at the Sagnac interferometer depicted very schematically in Figure 3.3. The output signal is the superposition of a beam which follows the dashed
path in the clockwise direction, with another which moves in the opposite direction, with a final phase shift given by

\[
\delta \phi_+ - \delta \phi_- \propto x_E(t) + x_N(t + \tau) - x_N(t) - x_E(t + \tau) \sim \tau [v_N(t) - v_E(t)]
\]

where in the last equality we discarded contributions proportional to higher powers of \( \tau d/dt \). This is a good approximation if we are interested in frequencies \( \Omega \ll \tau^{-1} \).

### 3.1.3 Modification of test mass dynamics

In a resonant cavity the cavity power is maximized, and in the linearized approximation radiation pressure does not change when the mirrors are moved. This means that informations about the signal is encoded only in the phase of the output beam.

Out of resonance, or in presence of a more complicated optical scheme (notably if a signal recycling mirror is present) this is no more true: part of the signal is present also in the intensity quadrature, and a restoring force proportional to the mirrors’ displacement is present ("optical spring").

The modified dynamics can be exploited to evade the standard quantum limit [7, 8]: however not all the optomechanical modes are stable, and a control strategy must be devised.
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3.1.4 Quantum feedback

It is possible to monitor the movement of a test mass with an auxiliary cavity. Using a carefully implemented feedback strategy, which exploits the light squeezing generated by ponderomotive effects, this test mass can be controlled in principle at the quantum level, subtracting completely radiation pressure noise [7, 8].

This subtraction does not reduce the gravitational signal because the measurement is done locally with a short cavity, insensitive to the wave’s strain. The effect of other sources of noise, like for example thermal ones, has been studied in [9]. It is also possible to devise a feedback strategy appropriate for a complete resonant cavity, using a single error signal [10]. A possible scheme is given is Figure 3.4. Starting from it we can write the length of the cavity A as

$$\Delta_{21} = \frac{1}{2} \hbar L_A + (\chi_2 g_2 - \chi_1 g_1) F + \Delta_{21}^{(th)}$$

$$+ \ h \left[ (\chi_1 + \chi_2) \xi_A \hat{a}_0 - \chi_2 \xi_B \hat{b}_0 \right]$$

(3.5)

where $\chi_i$ are the mechanical susceptibilities of the mirrors, $g_i$ the feedback gains and $\xi_K$ the optomechanical coupling of the $K$-th cavity which is given by

$$\xi_K^2 = \frac{8 \omega_0^{(K)} I_0^{(K)} F^{(K)}}{\hbar c^2}$$

The first term in Eq. (3.5) describes the change induced by the gravitational wave strain $\hbar$, and the second the action of the feedback force $F$. We inserted also a piece $\Delta_{21}^{(th)}$ proportional to the thermally induced motion.

The second row contains the motion induced by the radiation pressure fluctuations: $\hat{b}_0$ is the $\theta = 0$ quadrature of the field entering in the cavity B, $\hat{a}_0$ is the same for the field entering in the cavity A.
If we use as a feedback force an appropriate quadrature of the field exiting from the cavity B, which we can parameterize with the quadrature angle \( \theta \) as

\[
F = \hat{b}_\theta + 2\xi_B \Delta_{32} \sin \theta
\]

we can rewrite Eq. (3.5) in the form

\[
\Delta_{21} = \frac{1}{2} h (L_A + 2G\xi_B L_B \sin \theta)
+ \Delta_{21}^{(th)} + 2G\xi_B \Delta_{32}^{(th)}
+ G \sin \theta \hat{b}_{\pi/2}
+ [G(\cos \theta + 2\xi_B^2(\chi_2 + \chi_3) \sin \theta) - \chi_2 h\xi_B] \hat{b}_0
+ h\xi_A[(\chi_1 + \chi_2) - 2G\xi_B \chi_2 \sin \theta] \hat{a}_0
\]

or (3.6)

where

\[
G = \frac{\chi_2 g_2 - \chi_1 g_1}{1 - 2\xi_B \sin \theta(\chi_3 g_3 - \chi_2 g_2)}.
\]

From the first row we see that as we anticipated the effect of the feedback on the coupling of the cavity to the gravitational strain is negligible, as soon as \( L_A \gg L_B \).

By choosing appropriately the measured quadrature and the gain it is possible to set to zero the last two rows in Eq. (3.6), which contain the effect of radiation pressure fluctuations.

It comes out that when the mirrors behave as free masses the quadrature angle must be

\[
\cot \theta = \frac{3h\xi_B^2}{\mu \Omega^2}
\]

where \( \mu \) is a function of the mirrors’ masses. This kind of frequency dependent measurement can be implemented by filtering the output beam in a cavity with an appropriately chosen detuning, as the one showed in the figure. The gains must be also frequency dependent, more precisely the condition

\[
\frac{g_1 + g_2}{M_1 + M_2} - \frac{g_3}{M_3} = \frac{1}{2\xi_B} \sqrt{\Omega^4 + \frac{4h^2 \xi_B^4}{\mu^2}}
\]

must be satisfied.

To evaluate the sensitivity in presence of the feedback we observe that the measured output quadrature of the cavity A is given by \( \hat{a}_{\pi/2} + 2\xi_A \Delta_{12} \) and give us an estimator for the gravitational strain

\[
\hat{h} = h + \frac{2\Delta_{21}^{(th)} + 4\rho \Delta_{12}^{(th)} + 2\rho \xi_B^{-1}\hat{b}_{\pi/2} + \xi_A^{-1}\hat{a}_{\pi/2}}{L_A + 2\rho L_B}
\]

where \( \rho = (1 + M_1/M_2)/2 \).

From this expression we verify explicitly the complete cancellation of the back action forces. We can also evaluate the power spectrum of the quantum noise, which contain only shot noise contributions and is given by

\[
S_h = \frac{4}{(L_A + 2\rho L_B)^2} \left( \frac{1}{4\xi_A^2} + \frac{\rho^2}{\xi_B^2} \right).
\]

Note that the shot noise in the cavity B reduce the apparatus sensitivity through the feedback, so that in order to improve it we need to increase the field in both the cavities to increase the optomechanical couplings. Note also that the effect of thermal noise is increased by the feedback.
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3.2 Conclusion

The feasibility of an experimental study of these techniques is linked to the possibility of reducing up to a very high degree other source of noises, technical and fundamentals. An important issue is the production of states of light with high levels of squeezing. Until now the maximum level of squeezing obtained is around $7 \text{ dB}$, and the next challenge is fixed at $10 \text{ dB}$.

A connected point is the reduction of losses inside the apparatus, because a loss is equivalent to the introduction of a source of non squeezed quantum fluctuations.

The study of quantum noise is an active field of research. Until now many proposals for the evasion of the standard quantum limit has ben proposed at theoretical level, and we think that there is large space for new ideas.
Answers to exercises

**Exercise 1, page 10**

We evaluate explicitly the time derivative of the scalar product:

\[
\frac{d}{dt}(A_1, A_2) = \frac{i\epsilon_0 S}{\hbar} \left( \int A_1^* \frac{\partial}{\partial t} \left( \epsilon \frac{\partial A_2}{\partial t} \right) - A_2 \frac{\partial}{\partial t} \left( \epsilon \frac{\partial A_1^*}{\partial t} \right) \right) dx \\
+ \frac{i\epsilon_0 S}{\hbar} \int \epsilon \left( \frac{\partial A_1^*}{\partial t} \frac{\partial A_2}{\partial t} - \frac{\partial A_2}{\partial t} \frac{\partial A_1^*}{\partial t} \right) dx \\
= \frac{i\epsilon_0 S c^2}{\hbar} \left( \int A_1^* \frac{\partial}{\partial x} \left( \frac{1}{\mu} \frac{\partial A_2}{\partial x} \right) - A_2 \frac{\partial}{\partial x} \left( \frac{1}{\mu} \frac{\partial A_1^*}{\partial x} \right) \right) dx
\]

and if we integrate by part the last expression we get a boundary term

\[
\frac{d}{dt}(A_1, A_2) = \frac{i\epsilon_0 S c^2}{\hbar} \left[ \frac{1}{\mu} \left( A_1^* \frac{\partial A_2}{\partial x} - A_2 \frac{\partial A_1^*}{\partial x} \right) \right]
\]

which is zero if we use periodic boundary conditions.

**Exercise 2, page 12**

The solutions that reduce to an outgoing plane wave for \( x > 0 \) can be written as (\( k > 0 \))

\[
\tilde{f}_k = N_k \begin{cases} 
  e^{ikx} + r_x e^{-ikx} & x < 0 \\
  t_x e^{ikx} & x > 0
\end{cases}
\]

and the ones that reduce to an outgoing plane wave for \( x < 0 \) as (\( k < 0 \))

\[
\tilde{f}_k = N_k \begin{cases} 
  t_x e^{ikx} & x < 0 \\
  e^{ikx} + r_x e^{-ikx} & x > 0
\end{cases}
\]

Here \( N_k \) is a normalization constant that we can fix evaluating the scalar product

\[
(\tilde{f}_{k_1}, \tilde{f}_{k_2}) = \frac{\epsilon_0 S}{\hbar} (\omega_1 + \omega_2) \int_{-\infty}^{+\infty} \tilde{f}_{k_1}^* (x) \tilde{f}_{k_2} (x) dx.
\]

It is clear that only the even part of the integrand can give a contribution

\[
(\tilde{f}_{k_1}, \tilde{f}_{k_2}) = \frac{1}{2} \frac{\epsilon_0 S}{\hbar} (\omega_1 + \omega_2) \int_{-\infty}^{+\infty} \left( \tilde{f}_{k_1}^* (x) \tilde{f}_{k_2} (x) + \tilde{f}_{k_1}^* (-x) \tilde{f}_{k_2} (-x) \right) dx.
\]
Explicitly we get, when $k_1 k_2 > 0$,

$$
\langle \hat{f}_{k_1}, \hat{f}_{k_2} \rangle = \frac{1}{\hbar} \varepsilon_0 S (\omega_1 + \omega_2) N_k N_k \int_{-\infty}^{+\infty} e^{-i(k_1-k_2)x} + |t|^2 e^{i(k_1-k_2)x} + |t|^2 e^{-i(k_1-k_2)x} \, dx
$$

$$
+ \frac{1}{\hbar} \varepsilon_0 S (\omega_1 + \omega_2) N_k N_k \int_{-\infty}^{+\infty} r e^{i(k_1+k_2)x} + r e^{-i(k_1+k_2)x} \, dx
$$

the second row gives no contributions in this case and the final result is

$$
\langle \hat{f}_{k_1}, \hat{f}_{k_2} \rangle = \frac{4 \pi \varepsilon_0 S \omega_1}{\hbar} N_k^2 \delta(k_2 - k_1)
$$

When $k_1 k_2 < 0$ we get always zero. To obtain the canonical commutation relations

$$
[\hat{a}_{k_1}, \hat{a}_{k_2}^\dagger] = \delta(k_1 - k_2)
$$

we must choose

$$
N_k = \sqrt{\frac{\hbar}{4 \pi \varepsilon_0 S \omega}}.
$$

**Exercise 3, page 13**

We need to evaluate the matrix element

$$
\langle q - \frac{x}{2} \vert [\hat{H}, \hat{\rho}] \vert q + \frac{x}{2} \rangle
$$

where the Hamiltonian is given by $\hat{H} = \hat{p}^2 / 2M + U(\hat{q})$. For the kinetic term we have

$$
\langle q - \frac{x}{2} \vert \left[ \frac{\hat{p}^2}{2M}, \hat{\rho} \right] \vert q + \frac{x}{2} \rangle = \langle q - \frac{x}{2} \vert \frac{\hat{p}^2}{2M} \hat{\rho} - \hat{\rho} \frac{\hat{p}^2}{2M} \vert q + \frac{x}{2} \rangle
$$

$$
= -\frac{1}{2M} \left( \frac{\partial^2}{\partial(q - x/2)^2} - \frac{\partial^2}{\partial(q + x/2)^2} \right) \langle q - \frac{x}{2} \vert \hat{\rho} \vert q + \frac{x}{2} \rangle
$$

$$
= -\frac{1}{M} \frac{\partial^2}{\partial q \partial x} \langle q - \frac{x}{2} \vert \hat{\rho} \vert q + \frac{x}{2} \rangle \tag{4.1}
$$

while for the potential one

$$
\langle q - \frac{x}{2} \vert U(\hat{q}), \hat{\rho} \vert q + \frac{x}{2} \rangle = \langle q - \frac{x}{2} \vert U(\hat{q}) \hat{\rho} - \hat{\rho} U(\hat{q}) \vert q + \frac{x}{2} \rangle
$$

$$
= \left( U(q - \frac{x}{2}) - U(q + \frac{x}{2}) \right) \langle q - \frac{x}{2} \vert \hat{\rho} \vert q + \frac{x}{2} \rangle
$$

$$
= -i \sum_{n=0}^{\infty} \left( \frac{i}{2} \right)^{2k} \frac{1}{(2k+1)!} \frac{\partial^{2k+1} U(q)}{\partial q^{2k+1}} (-ixe)^{2k+1} \langle q - \frac{x}{2} \vert \hat{\rho} \vert q + \frac{x}{2} \rangle. \tag{4.2}
$$

Now if we substitute these expressions inside Equation 4.3, we obtain

$$
\frac{\partial W}{\partial t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx} \left[ -\frac{1}{M} \frac{\partial^2}{\partial q \partial x} - \sum_{n=0}^{\infty} \left( \frac{i}{2} \right)^{2k} \frac{1}{(2k+1)!} \frac{\partial^{2k+1} U(q)}{\partial q^{2k+1}} (-ixe)^{2k+1} \right] \langle q - \frac{x}{2} \vert \hat{\rho} \vert q + \frac{x}{2} \rangle \, dx \tag{4.3}
$$

and we can carry the first operator out of the integral integrating by parts, the second one expressing $x$ as a derivative $\partial / \partial p$ which act on the exponential.
**Exercise 4, page 14**

In this case we see that
\[
\phi_{-1}(q) = \hat{a}\phi_0(q) = \frac{1}{\sqrt{2}} \left( q + \frac{\partial}{\partial q} \right) \phi_0(q)
\]  
(4.4)
satisfy
\[
\hat{a}^+ \phi_{-1}(q) = \frac{1}{\sqrt{2}} \left( q - \frac{\partial}{\partial q} \right) \phi_{-1}(q)
\]
the solution is not normalizable
\[
\phi_{-1}(q) \propto \exp \left( \frac{q^2}{2} \right)
\]
neither it is \(\phi_0\), which is the solution of Eq. 4.4:
\[
\phi_0(q) \propto \exp \left( -\frac{q^2}{2} \right) \int_q^\infty e^{\xi^2} \, d\xi.
\]

**Exercise 5, page 14**

Using the definition we find
\[
W_n(p,q) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipx} \left\langle q - \frac{x}{2} \mid n \right\rangle \left\langle n \mid q + \frac{x}{2} \right\rangle \, dx
\]
and inserting the number eigenstate
\[
W_n(p,q) = \frac{1}{2^{n+1}\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left[ ipx - \frac{x^2}{4} \right] H_n \left( q - \frac{x}{2} \right) H_n \left( q + \frac{x}{2} \right) \, dx.
\]
Evaluating the integral we get
\[
W_n(p,q) = \frac{(-1)^n}{\pi} \exp \left( -(p^2 + q^2) \right) L_n \left( 2(p^2 + q^2) \right)
\]
where \(L_n\) is a Laguerre's polynomial. This function is plotted in Figure 4.1 on page 41 for some selected values of \(n\).

(a) Vacuum    (b) Single photon    (c) Two photons    (d) Three photons

Figure 4.1: Wigner’s functions for the number eigenstate in the particular cases \(n = 0, 1, 2, 3\).

First point to note is that these distributions depends only on \(p^2 + q^2\), which means that they do not contain any information about the phase. In other words in a number eigenstate (which is also an energy eigenstate) the phase is completely undetermined. Another important point is that \(W(p,q)\) can be negative. This is a signal of the non classical nature of the number eigenstates, which is not compatible with the existence of a probability distribution in the phase space.
Exercise 6, page 16
Differentiating Eq. (2.13) with respect the squeezing parameter we find that
\[
\frac{\partial \phi_0^0(q)}{\partial \zeta} = \frac{1}{2} (\hat{a}\hat{a} - \hat{a}^+\hat{a}^+) \frac{\partial \phi_0^0(q)}{\partial q} = \frac{1}{2} \left( \frac{1}{2} (\hat{a}\hat{a} - \hat{a}^+\hat{a}^+) \right) \phi_0^0(q)
\]
And integrating we find that
\[
\phi_0^0(q) = \hat{S}(\zeta) \phi_0(q), \quad \hat{S}(\zeta) = \exp \left[ \frac{\xi}{2} (\hat{a}\hat{a} - \hat{a}^+\hat{a}^+) \right].
\]

Exercise 7, page 18
The direction of the minor axis is clearly at \( \theta = \theta_s \). Here we recover the smaller fluctuation
\[
\Delta q_{\theta_s}^2 = e^{-2r_s}
\]
The ellipticity can be written as a function of the ratio between the minimum and the maximum fluctuation as
\[
e = \sqrt{1 - \frac{\Delta q_{\theta_s}^2}{\Delta q_{\theta_s}^2}} = \sqrt{1 - e^{-4r_s}}.
\]

Exercise 8, page 24
Starting from the relation
\[
\hat{b}(\omega) = T(\omega)\hat{a}(\omega)
\]
we can rewrite it for the two operators entering in the quadrature vector as
\[
\begin{align*}
\hat{b}_+ &= \hat{b}(\omega_0 + \Omega) = T(\omega_0 + \Omega) \hat{a}(\omega_0 + \Omega) = T_+ \hat{a}_+ \\
\hat{b}_- &= \hat{b}(\omega_0 - \Omega)^+ = T(\omega_0 - \Omega)^* \hat{a}(\omega_0 + \Omega)^+ = T_-^* \hat{a}_+
\end{align*}
\]
and using these equations we obtain directly
\[
\hat{b}(\Omega) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \hat{b}_+ + \hat{b}_-^t \\ -i\hat{b}_+ + i\hat{b}_-^t \end{array} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} T_+ \hat{a}_+ + T_-^* \hat{a}_+^+ \\ -iT_+ \hat{a}_+ + iT_-^* \hat{a}_+^+ \end{array} \right)
\]
\[
= \frac{1}{2\sqrt{2}} \left( \begin{array}{c} (T_+ + T_-^*) (\hat{a}_+ + \hat{a}_+^+) + i(T_+ - T_-^*) (-i\hat{a}_+ + i\hat{a}_+^+) \\ (T_+ + T_-^*) (-i\hat{a}_+ + i\hat{a}_+^+) + i(T_+ - T_-^*) (\hat{a}_+ + \hat{a}_+^+) \end{array} \right)
\]
\[
= \frac{1}{2} \left( \begin{array}{cc} T_+ + T_-^* & iT_+ - iT_-^* \\ iT_+ - iT_-^* & T_+ + T_-^* \end{array} \right) \frac{1}{\sqrt{2}} \left( \begin{array}{c} \hat{a}_+ + \hat{a}_+^t \\ -i\hat{a}_+ + i\hat{a}_+^t \end{array} \right)
\]
which is the requested result.

Exercise 9, page 27
The array represent a multivariate Gaussian probability distribution. Principal axes are directed in the direction of the eigenvectors, and their length is given by the square root of the eigenvalues. To determine these we must solve the characteristic equation
\[
\lambda^2 - (2 + K^2)\lambda + 1 = 0
\]
which gives
\[ \lambda_{\pm} = \frac{(2 + K^2) \pm \sqrt{(2 + K^2)^2 + 4}}{2} \]
while the eigenvectors can be written as
\[ \vec{u}_{\pm} = \begin{pmatrix} K \\ 1 - \lambda_{\pm} \end{pmatrix}. \]

Note that the product of the eigenvalues (the determinant of the matrix, its square root being proportional to the area of the ellipse) does not depend from \( K \). In the large \( K \) regime we have
\[ \sqrt{\lambda_{+}} \sim K \]
\[ \sqrt{\lambda_{-}} \sim K^{-1} \]
which means that the ellipse becomes very elongated (large squeezing).
This is an introductive quantum optics book which uses a formalism similar to the one described here.

Another introductive book, which is centered mainly on the theory of light detection. It contains a clear description of quasiprobability functions.


This is a basic reference text for the problem of quantum measurements and quantum non demolition. It is written by two pioneers of the field, and contain a large amount of informations and examples.


